Transfer Matrix Analysis of Cable-Stiffened Hoop Platforms

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A transfer matrix method is applied to the free-vibration analysis of a series of cable-stiffened hoop platforms, which are often encountered as subassemblies in large space antennas. The method takes advantage of the cyclic symmetry of the structures to reduce significantly the amount of computations necessary to determine natural frequencies and the associated mode shapes. Numerical difficulties, which often arise in transfer matrix analyses of periodic structures, are not encountered with the method presented. Free-vibration characteristics are calculated for equilateral hoop assemblies with from 5 to 11 segments. Properties are chosen to allow comparison with the earlier experiments and calculations of Belvin, and good agreement is shown for that specific, hexagonal hoop platform. Running time comparisons indicate that the transfer matrix approach provides a promising alternative to finite-element methods for the dynamic analysis of spacecraft structures characterized by slender substructures and repetitive geometry.

Nomenclature

A = cross-sectional area of a hoop segment
$[A]$ = transformation of axes transfer matrix
$[C]$ = cable transfer matrix
e = 2.71828
e_v = in-plane mass offset
e_w = out-of-plane mass offset
$[E]$ = elastic member transfer matrix
EA = axial stiffness of a hoop segment
$(EA/L)_c$ = cable stiffness
EI = bending stiffness of a hoop segment
F = axial force, state variable
GJ = torsional stiffness of a hoop segment
i = square root of -1
$[I]$ = identity matrix
I_v = in-plane lumped mass moment of inertia
I_{vw} = crossproduct mass moment of inertia
I_w = out-of-plane lumped mass moment of inertia
I_θ = torsional mass moment of inertia
j = index on hoop joints
k = index on point masses for each segment
k_c = cable stiffness $= (EA/L)_c$
K_v = in-plane shear form factor
K_w = out-of-plane shear form factor
L = hoop segment length
L_c = cable length
m = lumped mass
M = total mass of one hoop segment
$[M]$ = mass transfer matrix
M_v = in-plane bending moment, state variable
M_w = out-of-plane bending moment, state variable
N = number of hoop segments
NS = number of point masses in discretization
P = hoop segment preload (compression)
P_c = cable preload (tension)
$[s]$ = hoop joint state vector at a given value of ω
$[\bar{s}]$ = hoop joint state vector having displacements and slopes orthogonal to those of $[s]$ at the same value of ω
T = torque, state variable
$[T]$ = matrix relating two adjacent hoop joint state vectors
u = axial displacement, state variable
v = in-plane transverse displacement, state variable
V = in-plane shear force, state variable
w = out-of-plane transverse displacement, state variable
W = out-of-plane shear force, state variable
$[z]$ = hoop joint state vector
α = transformation of axes rotation angle
β = angle between cable and normal to hoop plane
γ = phase angle between two orthogonal modes occurring at the same value of ω
δ = angle between cable and hoop vertex bisector (measured in the plane of the hoop)
θ = twist, state variable
λ = nondimensional segment preload parameter
ϕ = in-plane slope, state variable
ψ = out-of-plane slope, state variable
ω = natural frequency of the hoop
$\bar{\omega}$ = nondimensional natural frequency of the hoop

Introduction

THE structural dynamics of large space systems such as orbiting satellite antennas and solar power generators are important to their stability and proper functioning. It is usually necessary to determine the free-vibration frequencies and modes of large space structures with considerable accuracy and efficiency to ensure, for example, that corrective impulses, as applied by vernier thrusters or control gyros, will not inadvertently trigger unstable closed-loop oscillations. In addition, phenomena such as gravity gradients, orbital dynamics, and cyclic heating through penumbral excitation may serve as sources of large-amplitude vibrations.

The mathematical models that result from the straightforward application of such powerful approaches as the finite-element method to the dynamic analysis of these structures can prove costly. Often, simpler structures possessing similar attributes or structural assemblies that are parts of the whole are analyzed to determine general dynamic characteristics and trends.

The transfer matrix method can usually be applied to beamlike structures so as to provide as much accuracy as a full NASTRAN formulation, but with efficiencies sufficient to serve as a practical design tool. This paper describes an application of the transfer matrix method to the determination of the free-vibration behavior of a series of cable-stiffened hoop platforms. Hexagonal spacecraft subassemblies of this

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type were studied by Belvin[1] both experimentally and using a NASTRAN analysis. Both in-plane and out-of-plane vibrations are examined here for hoops possessing from 5 to 11 segments. In this application, maximum advantage is taken of the repetitive geometry of the structures.

Model

The general hoop model consists of an arbitrary number of straight, rectangular beams connected rigidly to each other, i.e., with moment and torque transfer at their connecting points, so as to form an equilateral polygonal frame. A hexagonal configuration is shown in Fig. 1. Pretensioned, diametral cables are attached at the joints between segments to increase overall frame stiffness. The hoop segments are assumed to undergo oscillations in out-of-plane and in-plane bending, axial compression-extension, and torsion, all small enough such that Euler-Bernoulli beam and St. Venant torsion theories apply and so that the cable pretension is never overcome. The analysis assumes that the cable mass is negligible.

A lumped parameter discretization of the structure is performed in which each hoop segment is represented as a series of rigid masses of infinitesimal geometrical dimensions but possessing mass moment of inertia properties connected by massless elastic one-dimensional beam-column-torsion members.

Analysis Procedure

The long thin nature of the hoop segments suggests that a transfer matrix approach[2] will provide computational advantages. The essence of the method is to relate state variables successively, i.e., transverse shear forces, bending moments, bending slopes, and transverse deflections—all in-plane and out-of-plane—axial force and axial deflection, and torsional moment and twisting deflection, from one location in the structure to another through multiplication of transfer matrices.

The collection of dependent variables at a point on the structure, referred to as a state vector, is transferred successively across the point masses and elastic elements of a segment. These transfers are accomplished using mass transfer matrices, whose elements are obtained through consideration of dynamic equilibrium relations, and elastic transfer matrices, whose elements are derived based on subsidiary static analyses, which establish the beam-equivalent quantities.

Where the structure has a bend, as at the joints connecting the segments, a rotational transfer matrix is used in the multiplication scheme, which amounts simply to a transformation of beam axes. The transfer across a hoop joint also must account for the change in forces at the cable attachment points due to cable stretching and redirection of the load of the pretensioned cable. Such transfers are performed subsequent to rotating the state vector at the end of a hoop segment through one-half of the angle separating the axis systems of adjacent segments. General forms for all of these transfer matrices are given in the Appendix. Note that, for the hoop model shown in Fig. 1, the cable transfer matrix [as defined in the Appendix, Eq. (A6)] simplifies to the following uncoupled forms for in-plane and out-of-plane motion:

In-plane case:

$$\begin{bmatrix} V \\ M_v \\ \phi \\ v \\ F \\ u \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & (EA/L)_c & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} V \\ M_v \\ \phi \\ v \\ F \\ u \end{bmatrix}$$

Out-of-plane case:

$$\begin{bmatrix} W \\ M_w \\ \psi \\ w \\ T \\ \theta \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & P_c/L_c & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} W \\ M_w \\ \psi \\ w \\ T \\ \theta \end{bmatrix} \tag{1}$$

As a consequence of the rotational periodicity of the hoop, transfer around the complete hoop (in either direction) is unnecessary. Using Thomas' method of cyclic symmetry,[3] it is possible to determine the natural frequencies and mode shapes of the whole structure by considering the relationship between the modal behavior of the smallest repeating substructure (i.e., one hoop segment including half-cables at either end) and the remaining substructures. This is done by defining a complex hoop joint state vector by linearly combining two sets of state variables corresponding to modes that are orthogonal and have the same natural frequency. That is,

$$[z]_j = [s]_j + i[\bar{s}]_j \tag{2}$$

The periodicity of closed, cyclically symmetric bodies makes it possible to relate adjacent hoop joint state vectors as follows:

$$[z]_j = e^{i\gamma}[z]_{j-1} \tag{3}$$

Since the application of Eq. (3), N times, where N is the number of repeating substructures (the number of hoop segments), must leave $[z]_j$ unchanged (i.e., the structure closes on itself), γ must assume one of the values

$$\gamma = 2\pi n/N \tag{4}$$

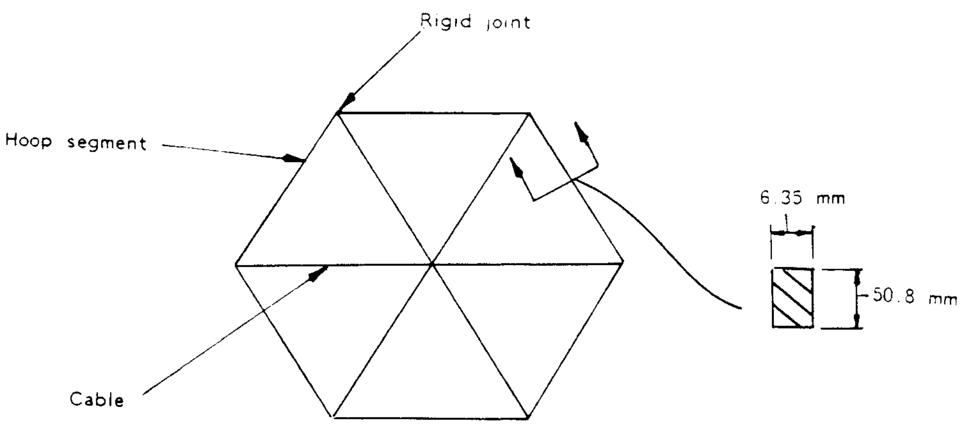

Fig. 1 Hexagonal, cable-stiffened hoop platform (plan view).

where n is an integer. The phase angle γ may have $(N/2 + 1)$ possible independent values for N even and $(N + 1)/2$ values if N is odd. These are:

$$0,\ 2\pi/N,\ 4\pi/N,\ \dots,\ 2\pi(N/2-1)N,\ \pi \tag{5}$$

for N even, and

$$0,\ 2\pi/N,\ 4\pi/N,\ \dots,\ 2\pi(N-1)/2N \tag{6}$$

for N odd.

It follows that the conditions for free vibrations are arrived at by first transferring between adjacent hoop joints according to the following matrix multiplication scheme:

$$[z]_j = [T][z]_{j-1}$$

where

$$[T] = \{[I]-[C]\}^{-1}[A]\left(\underset{k=1}{\overset{NS}{\pi}}[E]_k[M]_k\right)[A]\{[I]+[C]\} \tag{7}$$

In this relation, the $[A]$ matrices accomplish the transfer through one-half of the angle separating adjacent segment axis systems, the $[C]$ matrix [derived in the Appendix and displayed in simplified form in Eq. (1)] accounts for cable effects, the symbol π indicates a successive matrix product, and NS is the number of point-mass/elastic member combinations used in the discretization of one hoop segment.

At this point, effective boundary conditions may be enforced by choosing a value of γ and requiring the vectors $[z]_j$ and $[z]_{j-1}$ to satisfy Eq. (3). This can be expressed as

$$\{(\cos\gamma + i\sin\gamma)[I] - [T]\}[z]_j = 0$$

or

$$[D][z]_j = 0 \tag{8}$$

For the above relation to have a nontrivial solution, $[z]_j \neq 0$, the determinant of $[D]$ must vanish. The values of ω that make $[D] = 0$ correspond to the natural frequencies of the full hoop.

The solution procedure is one of trial and error, in which all of the frequencies of the structure may be determined by locating the zeros of the complex determinant of $[D]$ for each possible value of γ. Mode shapes are obtained by solving Eq. (8) for $[z]_j$ (at a particular ω) and repeating the transfer process, Eq. (7), from station to station. For modes in which the imaginary part of $[z]_j$ is not zero (i.e., those modes for which $\gamma \neq 0, \pi$), the transfer process may be repeated twice; using first the real part of $[z]_j$ and then the imaginary part, to find each of the two orthogonal modes possessing the same eigenvalue.

It may not be obvious that use of cyclic symmetry relations reduces the possibility that a mode may be "missed" in a trial frequency search procedure. Large flexible structures typically possess many closely spaced modes of different types (i.e., modes that may be described with different values of γ). Determining natural frequencies by a method that deals with each of the possible values of γ separately should result in better resolution of the closely spaced modes of such structures.

Results and Discussion

In the specific cases analyzed here, a hoop radius equal to 1 m is maintained, as the number of hoop segments is varied. This hoop size as well as the segment and cable properties shown in Table 1 were deliberately chosen to facilitate comparison with the results of Belvin.[1] Note that cable preload is assigned values such that the ratio of the segment preload to its Euler buckling load is always equal to 0.01 for in-plane vibrations and to 0.002 for the out-of-plane cases for each hoop configuration. Note that hoop natural frequencies may be nondimensionalized by dividing by the first natural bending frequency of a simply supported beam with zero axial load whose characteristics are otherwise those of a hoop segment. This frequency is as follows:

$$\bar{\omega}^2 = \omega^2 L^3 M/\pi^4 EI \tag{9}$$

where either the in-plane or out-of-plane value of EI is used, as appropriate.

In-plane and out-of-plane natural frequencies were obtained for hoops with 5 through 11 sides. Table 2 compares the present results and those of Belvin. Figure 2 shows the effect of altering the number of segment sides on the first six in-plane and out-of-plane natural frequencies of the 1 m radius hoop. Note the "crossover" that makes the nominal fifth in-plane mode frequency drop beneath those of the nominal third and fourth modes when the number of segments increases from 8 to 10. The in-plane frame stiffening brought about by raising the number of segments (and, thus, cables) of the constant diameter hoop is demonstrated in Fig. 2a, where a fairly steady rise in the first six in-plane frequencies can be observed as the number of hoop segments is increased. Since

Table 1 Hoop segment and cable properties

Hoop segments	
Mass/length, kg/m	0.8755
Segment cross-sectional A, $\times 10^{-4}$ m^2	3.2258
EI in-plane, N-m^2	76.96
EI out-of-plane, N-m^2	4925
EA, MN	22.90
GJ, N-m^2	113.3
Cables	
Cable length, m	1
Stiffness, $\times 10^5$ N	1.693
Preload (in-plane case),[a] N[b]	7.5954
Preload (out-of-plane),[c] N[b]	97.224

[a]In-plane cable preload varies such that the ratio of the resulting compressive force in each hoop segment to its Euler buckling load remains at 0.01 as the number of hoop sides is varied.
[b]Values shown correspond only to the hexagonal case.
[c]Out-of-plane cable preload varies such that the ratio of resulting compressive force in each hoop segment to its Euler buckling load remains at 0.002 as the number of hoop sides is varied.

Table 2 Comparison between NASTRAN and transfer matrix calculation results for the natural frequencies of the hexagonal hoop

Mode	TM method	NASTRAN[1]
	In plane	
1	92.1	92.1
2	116.5	116.6
3	162.0	—[a]
4	210.0	210.1
	Out of plane	
1	59.00	58.41
2	129.00	128.10
3	740.00	740.30
4	955.00	947.60
5	1369.00	1355.00

[a]Not reported in Belvin's results, probably due to a difference in modeling of the cable junction at the "origin" or center of the hexagon, which shows appreciable motion in this mode.

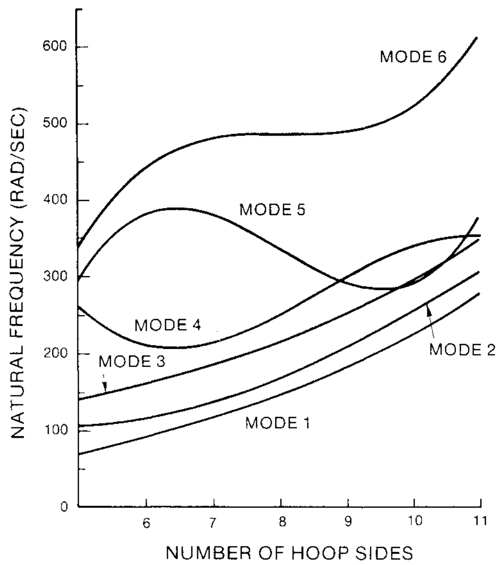

Fig. 2a Variation of the first six in-plane modes with increasing hoop segment number.

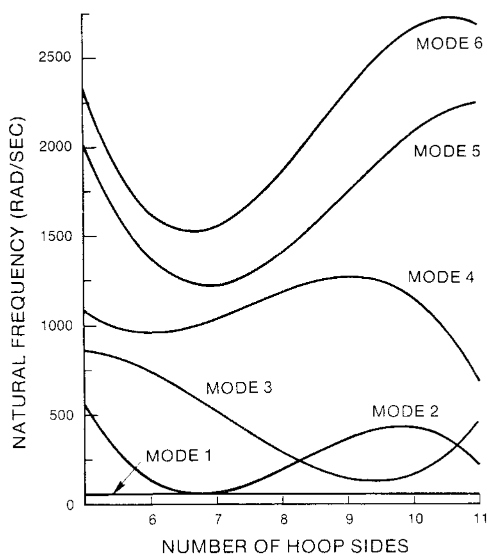

Fig. 2b Variation of the first six out-of-plane modes with increasing hoop segment number.

this analysis assumes first-order small displacements, out-of-plane vibrations are influenced significantly less by cable stiffness.

Mode shapes were calculated for 5-, 6-, 8-, and 11-sided hoops. Figures 3 and 4 illustrate the first three mode shapes of each hoop configuration for in-plane and out-of-plane vibrations, respectively. Figure 3, presenting in-plane modes, views the hoop in planform; the out-of-plane modes, in Fig. 4, are shown isometrically. Two mode shapes are displayed at the same frequency only where differences in mode shape were clearly discernible. It should be noted that equilibrium was checked and verified for such modes.

Significant amounts of beam bending are present in the first several in-plane modes. The first in-plane mode for both the hexagonal and octagonal hoops is, in fact, the first simply supported beam bending mode of the segments. A first cantilever mode arises as the third in-plane mode of the pentagonal platform.

Torsion dominates the first two out-of-plane modes, with only the second mode of the five-sided frame exhibiting any appreciable beam bending. Unfortunately, diagrams such as those in Fig. 4 do not show torsion explicitly. The effect of torsion can be discerned, however, as a discontinuous change of bending slope from one segment to the next at the vertices or at joints between segments. The third mode of the hexagonal hoop is seen to be the only simply supported beam mode present in the lower out-of-plane modes of the hoops.

For the purpose of comparison with presently used methods of numerical analysis, such as the finite-element method, execution times for the computation of a value of the determinant of the matrix $[D]$, as defined in Eq. (8), and for the calculation of mode shapes of the hexagonal hoop (only in-plane motion was considered for the timing measurements), have been noted with respect to similar execution times obtained by Belvin using NASTRAN.[1] The computations reported here were performed on an IBM 3081-D using the Michigan Terminal System operating system, Version 5.1C. Belvin's computations were performed on a Cyber 855. Comparison between the two machines' running times was provided by MacNeil Schwendler Company (the distributors of MSC NASTRAN), who reported that the 855 requires less time than the 3081-D in the ratio 1.3:3.4, in terms of CPU time for multiplication-add loops, when running NASTRAN.

In the transfer matrix analysis, each hoop segment was modeled with 18 "nodes" (i.e., 6 point masses each with 3 degrees of freedom), for a total of 108 degrees of freedom for the hexagon. In comparison, Belvin's hexagonal hoop model contained 43 two-degree-of-freedom nodes, for a total of 86 degrees of freedom. Eigenvalue determinations for the present model were made, as mentioned previously, through a trial-and-error process. Belvin determined the 86 eigenvalues of his model using the program GIVENS.

For each trial frequency, the calculation of the determinant of the matrix D [Eq. (8)] required 0.095 CPU seconds. While the number of trials needed to find a natural frequency varied substantially, 5–10 trials were needed on the average. It seems likely that a mechanized frequency search procedure (i.e., a root-finding algorithm) would reduce the required number of evaluations of the determinant of $[D]$. In comparison, GIVENS required 39 CPU seconds to find the 86 eigenvalues of Belvin's hoop model.

The natural frequency efficiency comparison, for say, 10 modes in terms of Cyber 855 speeds would, therefore, be (0.095 CPU seconds/trail) (7.5 trials) (10 modes) (1.3/3.4) = 2.72 s/10 modes for the transfer matrix approach, and (39 CPU seconds/86 modes) 1/8.6 = 4.53 s/10 modes for the NASTRAN analysis. It is recognized that NASTRAN, in general, possesses a cyclic symmetry capability that GIVENS does not utilize. The preceeding comparison, however, was the only one available to the authors.

The determination of a single mode shape averaged 0.885 CPU seconds with the present method of analysis. NASTRAN required 325 CPU seconds to calculate all of the eigenvectors and to perform the necessary plotting. On a similar basis, the 10-mode-shape running time comparison is (0.885 CPU seconds/modes) (10 modes) (1.3/3.4) = 3.38 s/10 modes for the transfer matrix analysis, as compared with (325 CPU seconds/86 modes) 1/8.6 = 37.8 s/10 modes for the NASTRAN analysis. It should be pointed out that the CPU times referred to here are problem state (PS) CPU times, which are understood to be the execution times if the machine were idle and empty except for the tasks of interest.

It is apparent from the preceding that the transfer matrix method, coupled with cyclic symmetry relations, is an efficient analysis alternative to NASTRAN for analyzing the free-vibration behavior of a cable-stiffened polygonal hoop. While it is certainly not contended that the transfer matrix method is generally superior to NASTRAN or other finite-element techniques for the dynamic analysis of space structures, specialized transfer matrix analysis programs do have some advantages. These include reduced storage requirements (i.e., the order of the transfer matrices of the model are determined by the number of degrees of freedom at any one station) and the straightforward manner in which modal properties may be determined. Thus, the transfer matrix method does seem to remain a useful tool for the dynamic analysis and design of those large, complex space structures characterized by slender substructures and repetitive geometry.

Problem of Numerical Difficulties

There are no theoretical reasons why an analysis of cyclically symmetric structures could not be performed without invoking this symmetry and simply transferring completely "around" the system, i.e., successively across all of the hoop segments. The requirement for the structure to "close" on itself leads to the matrix form of the characteristic equation of the system. For example, by first calculating the matrix $[T]$ [defined in Eq. (7)] to the Nth power, where N is the number of hoop segments, the following relation may be written:

$$[z]_{j+N} = [T]^N[z]_j \tag{10}$$

However, since $[z]_{j+N}$ must equal $[z]_j$ Eq. (10) may be rewritten as

$$\{[I] - [T]^N\}[z]_j = 0 \tag{11}$$

For $[z]_j$ to be nonzero, the determinant of $\{[I] - [T^N]\}$ must vanish for trial frequencies corresponding to the natural frequencies of the hoop.

While this approach seems straightforward when compared with an analysis that takes advantage of the structure's periodicity, numerical problems arise in the formation of the matrix $[T]^N$. As discussed by Uhrig[4] and Davies and Dawson,[5] calculating the product of transfer matrices in a long chain of identical components (in this case, the hoop segments with half-cables at each end) may lead to large errors, because all of the columns of the matrix product $[T]^N$ tend to be parallel with the eigenvector associated with the dominant eigenvalue of the component matrix ($[T]$), as N becomes large. Note that this is inherent in the iteration technique for calculating characteristic values.

A method suggested by Davies and Dawson to avoid the parallelism phenomena involves the creation of a "supermatrix" that contains blocks of $[T]$ along its diagonal. This large matrix is then used to obtain $[T]^N$ by Gaussian elimination. While this approach may be essentially free of numerical difficulties for nonclosed periodic structures, problems still arose, in the form of erratic behavior of the determinant values, when this "supermatrix" scheme was applied to the hoop model. This suggests that there are some further inherent numerical difficulties present in the application of the transfer matrix method to closed, cyclically symmetric structures. By taking advantage of the symmetry of the structures, it is possible to avoid them (as demonstrated by the previous analysis).

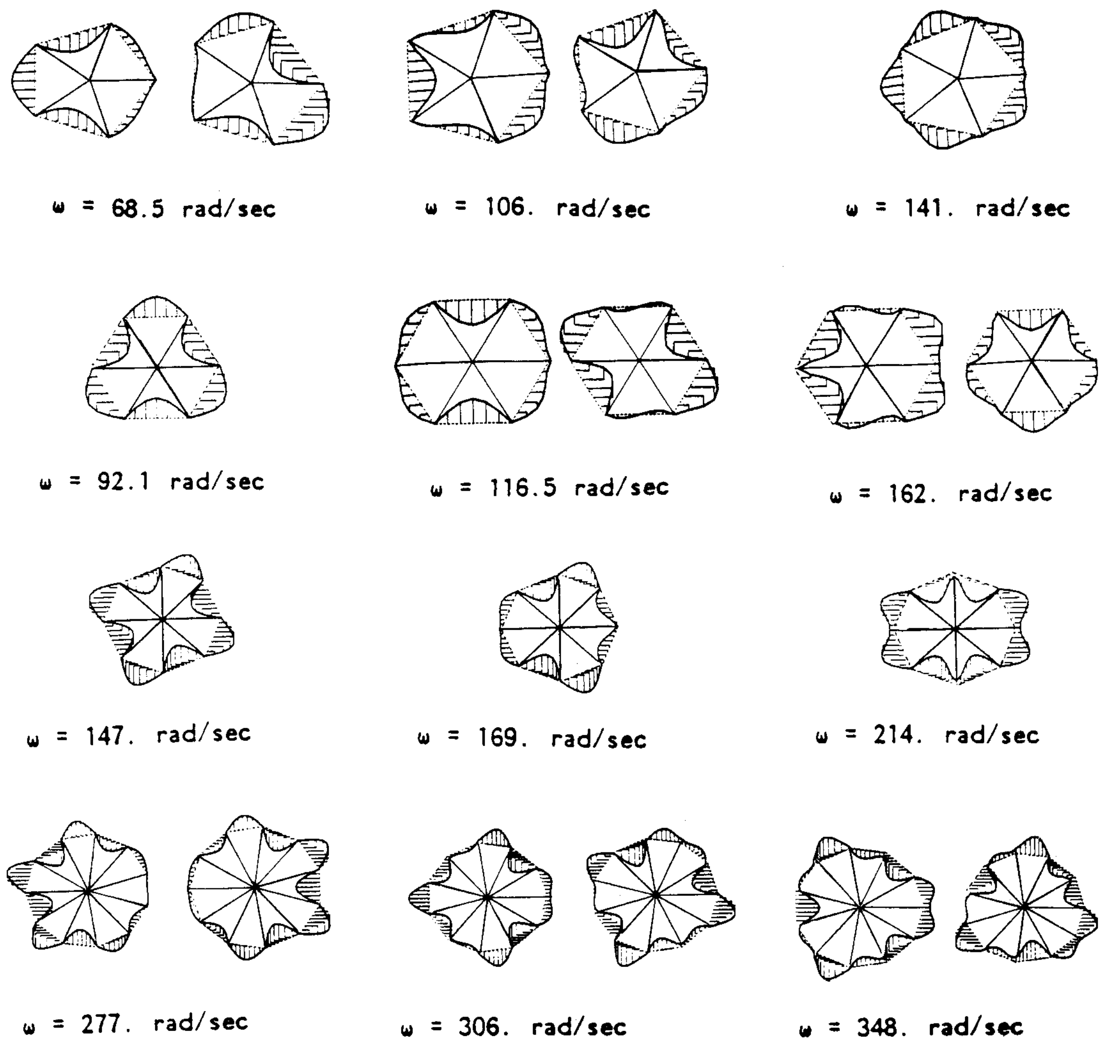

Fig. 3 First three in-plane mode shapes of the 5-, 6-, 8-, and 11-sided hoops.

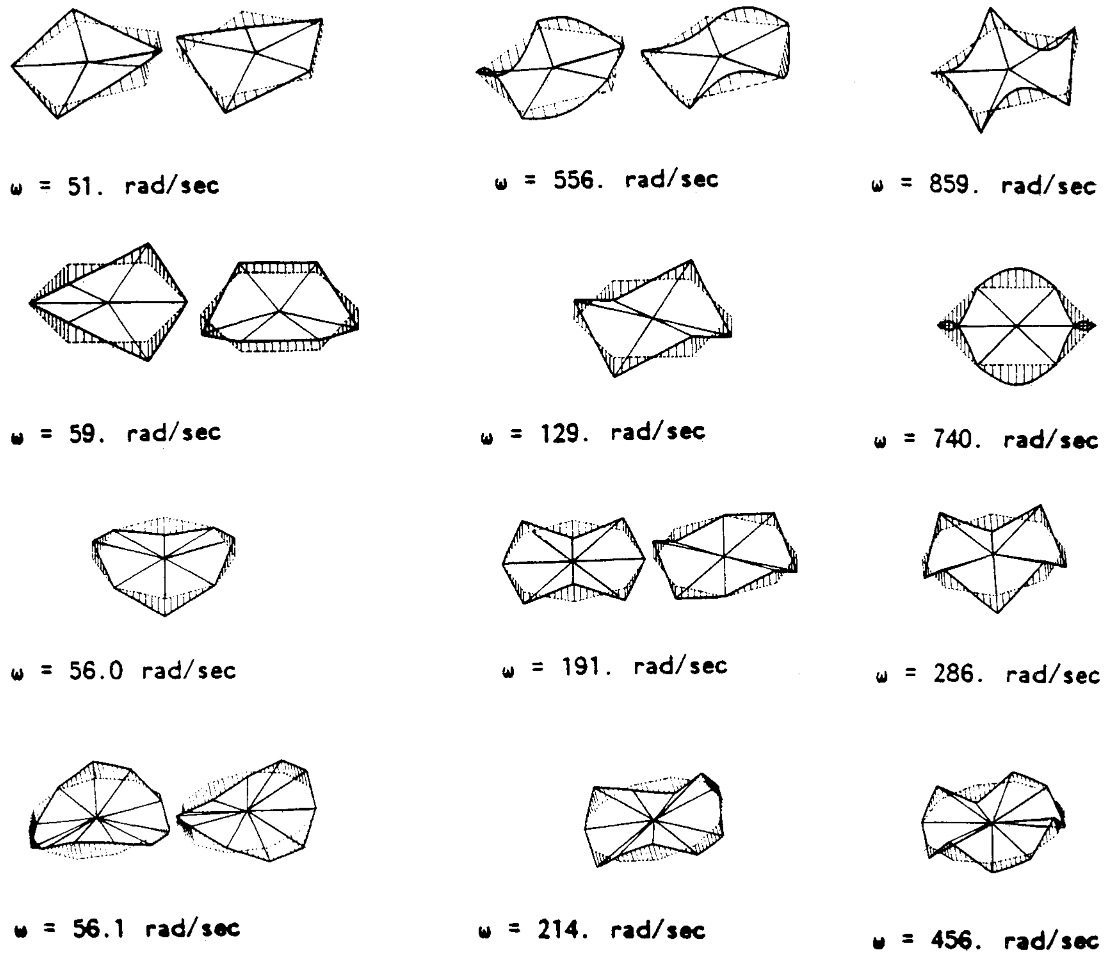

Fig. 4 First three out-of-plane mode shapes of the 5-, 6-, 8-, and 11-sided hoops

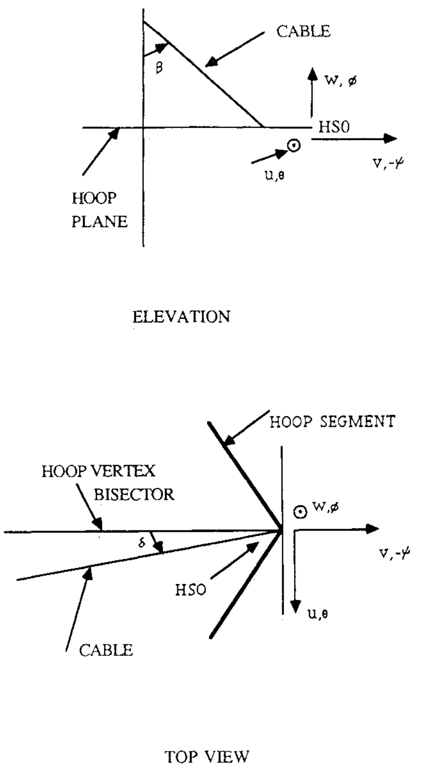

Fig. A1 Definition of hoop joint degrees of freedom and cable angles.

Concluding Remarks

A method combining transfer matrices and the concept of rotational periodicity (or cyclic symmetry) has been applied to the free-vibration analysis of a series of cable-stiffened, pretensioned hooplike structural assemblies often considered as subassemblies for satellite antennas. A high degree of computational efficiency is achieved by taking full advantage of the periodicity of the structure. Further, required storage is reduced, since the order of the matrices to be manipulated is immune to the number of degrees of freedom of the system under investigation—the transfer matrix size being fixed by the problem type, i.e., four state quantities for bending in a given direction: two for axial compression/tension and two for torsion. Timing comparisons between the subject method and a similar NASTRAN analysis for a hexagonal hoop suggest that transfer matrix methods may provide an efficient alternative numerical tool for spacecraft dynamic analysis. Integration of the technique presented here into transfer matrix analyses of the more complex free-vibration behavior of entire large, sparse, flexible, space structures is seen as promising.

Appendix

Mass, elastic member, rotation, and cable transfer matrices are presented here, which are general in the sense that they apply for the case of fully coupled bending in two directions (in-plane and out-of-plane designations are arbitrary), axial compression/tension and torsion. The matrices are displayed in a 12×12 form, which would be needed, for example, in a more general analysis of a complete cable-stiffened antenna. Derivations of the elastic, mass, and rotation matrices can be found in Ref. 2. The cable transfer matrix is derived here.

State variables across a lumped mass can be related as

$$
\begin{bmatrix} V \\ M_v \\ \phi \\ v \\ W \\ M_w \\ \psi \\ w \\ F \\ u \\ T \\ \theta \end{bmatrix}_L =
\begin{bmatrix}
1 & 0 & 0 & \omega^2 m & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\omega^2 me_w \\
0 & 1 & -\omega^2 I_v & 0 & 0 & 0 & -\omega^2 I_{vw} & 0 & 0 & \omega^2 me_v & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & \omega^2 m & 0 & 0 & 0 & \omega^2 me_v \\
0 & 0 & -\omega^2 I_{vw} & 0 & 0 & 1 & -\omega^2 I_w & 0 & 0 & \omega^2 me_w & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & -\omega^2 me_v & 0 & 0 & 0 & -\omega^2 me_w & 0 & 1 & \omega^2 m & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & -\omega^2 me_w & 0 & 0 & 0 & \omega^2 me_v & 0 & 0 & 1 & \omega^2 I_\theta \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix} V \\ M_v \\ \phi \\ v \\ W \\ M_w \\ \psi \\ w \\ F \\ u \\ T \\ \theta \end{bmatrix}_R
$$

where the coupling between the various classes of motion arises due to the presence of a mass crossproduct of inertia terms and mass offsets from the elastic axis along the in-plane and out-of-plane directions. The R and L designations refer to the right- and left-hand sides of the mass, respectively.

A transfer across an elastic member preloaded in compression may be performed as follows:

$$
\begin{bmatrix} V \\ M_v \\ \phi \\ v \\ W \\ M_w \\ \psi \\ w \\ F \\ u \\ T \\ \theta \end{bmatrix}_L =
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{L \sin\lambda_v}{\lambda_v} & \cos\lambda_v & \frac{-PL \sin\lambda_v}{\lambda_v} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1-\cos\lambda_v}{P} & \frac{\lambda_v \sin\lambda_v}{PL} & \cos\lambda_v & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{L}{P}\left(1-\frac{\sin\lambda_v}{\lambda_v}\right)-\frac{K_v L}{GA} & \frac{1-\cos\lambda_v}{P} & \frac{L \sin\lambda_v}{\lambda_v} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{L \sin\lambda_w}{\lambda_w} & \cos\lambda_w & -\frac{PL \sin\lambda_w}{\lambda_w} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1-\cos\lambda_w}{P} & \frac{\lambda_w \sin\lambda_w}{PL} & \cos\lambda_w & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{L}{P}\left(1-\frac{\sin\lambda_w}{\lambda_w}\right)-\frac{K_w L}{GA} & \frac{1-\cos\lambda_w}{P} & \frac{L \sin\lambda_w}{\lambda_w} & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{-L}{EA} & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{-L}{GJ-PK_a} & 1
\end{bmatrix}
\begin{bmatrix} V \\ M_v \\ \phi \\ v \\ W \\ M_w \\ \psi \\ w \\ F \\ u \\ T \\ \theta \end{bmatrix}_R
$$

where the term $\lambda = (PL^2/EI)^{1/2}$.

An axis transformation, as in rotating between adjacent hoop segments with axis angular misalignments, may be represented as

$$
\begin{bmatrix} V \\ M_v \\ \phi \\ v \\ W \\ M_w \\ \psi \\ w \\ F \\ u \\ T \\ \theta \end{bmatrix}_L =
\begin{bmatrix}
\cos a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\sin a & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \cos a & 0 & 0 & 0 & 0 & 0 & -\sin a & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \cos a & 0 & 0 & 0 & 0 & -\sin a & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \cos a & 0 & 0 & 0 & 0 & \sin a \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
\sin a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cos a & 0 & 0 & 0 \\
0 & 0 & 0 & \sin a & 0 & 0 & 0 & 0 & 0 & \cos a & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \sin a & 0 & 0 & 0 & 0 & \cos a & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -\sin a & & 0 & 0 & 0 & \cos a
\end{bmatrix}
\begin{bmatrix} V \\ M_v \\ \phi \\ v \\ W \\ M_w \\ \psi \\ w \\ F \\ u \\ T \\ \theta \end{bmatrix}_R
$$

where α is the rotation angle.

To derive the cable transfer matrix, which accounts for the cable forces at a hoop joint, the angles δ and β and the degrees of freedom at the hoop joint are first defined as shown in Fig. A1. The angles β and δ are zero in all of the cases examined in this paper, but may not be zero when a hoop platform is incorporated as a substructure of a larger spacecraft. Projections of the unstretched cable length, L_c, onto the hoop coordinate axes can then be written as

for the x direction,

$$L_c \sin\delta \sin\beta \tag{A1a}$$

for the y direction,

$$L_c \cos\delta \sin\beta \tag{A1b}$$

for the z direction,

$$-L_c \cos\beta \tag{A1c}$$

Thus, the amount of stretch in a cable is

$$[\{L_c \cos\beta - w_{HS}\}^2 + \{L_c \cos\delta \sin\beta + v_{HS}\}^2 + \{L_c \sin\delta \sin\beta + - u_{HS}\}^2]^{1/2} - L_c \tag{A2}$$

Where the subscript HS identifies the point where the cable is connected to the hoop vortex.

Dropping second-order terms in the small displacements u, v, w, ϕ, ψ, and θ, leaves for the cable stretch,

$$L_c\{[1 + 2/L_c(-u_{\mathrm{HS}}\sin\delta\sin\beta + v_{\mathrm{HS}}\cos\delta\sin\beta - w_{\mathrm{HS}}\cos\beta)]^{1/2} - 1\}$$

Using the binomial theorem to expand $(1+a)^{1/2} = 1 + \frac{1}{2}a$ + higher-order terms (when a is much less than 1) allows writing as an approximation for the cable stretch,

$$-u_{\mathrm{HS}}\sin\delta\sin\beta + v_{\mathrm{HS}}\cos\delta\sin\beta - w_{\mathrm{HS}}\cos\beta = \Delta L_c \tag{A3}$$

Similarly, the changes in the direction of the cable preload, P_c, must be taken into account since a cable-hoop segment joint may be in equilibrium before deflection but may not be after deformation, even if the force magnitudes are unchanged.

Taking advantage of the small angle approximations, the angle changes may be written as follows (see Fig. A1):

for β,

$$\Delta\beta_i = 1/L_c[w_{\mathrm{HS}}\sin\beta + v_{\mathrm{HS}}\cos\beta\cos\delta - u_{\mathrm{HS}}\cos\beta\sin\delta] \tag{A4}$$

for δ,

$$\Delta\delta = [-w_{\mathrm{HS}}\cos\delta - v_{\mathrm{HS}}\sin\delta]1/L_c\sin\beta$$

Total cable forces, assuming a cable spring rate of k_c, are

$$U + \Delta U = (P_c + k_c\,\Delta L_c)\sin(\beta+\Delta\beta)\sin(\delta+\Delta\delta)$$

$$V + \Delta V = -(P_c + k_c\,\Delta L_c)\sin(\beta+\Delta\beta)\cos(\delta+\Delta\delta)$$

$$W + \Delta W = (P_c + k_c\,\Delta L_c)\cos(\beta+\Delta\beta)$$

where

$$U = P_c\sin\beta\sin\delta$$

$$V = P_c\sin\beta\cos\delta$$

$$W = P_c\cos\beta$$

Changes in forces acting on the hoop joint can then be written as

in the x direction,

$$\Delta U = -k_c\Delta L_c\sin\beta\sin\delta + P_c[\Delta\beta\sin\delta\cos\beta + \Delta\delta\cos\delta\sin\beta]$$

in the y direction,

$$\Delta V = -k_c\Delta L_c\sin\beta\cos\delta - P_c[\Delta\beta\cos\delta\cos\beta - \Delta\delta\sin\delta\sin\beta]$$

in the z direction,

$$\Delta W = k_c\Delta L_c\cos\beta - P_c[\Delta\beta\sin\beta\,\Delta\beta] \tag{A5}$$

If Eqs. (A3) and (A4) are inserted into Eq. (A5), it is possible to write a matrix equation for the transfer "across" a cable attachment at a hoop joint. This will be in the form

$$[z]_{j_L} = ([I] + [C])\,[z]_{j_R} \tag{A6}$$

where $[I]$ is the unit matrix, and the matrix $[C]$ accounts for the cable forces acting on a hoop joint due to hoop joint motion. The matrix form of $[C]$ can be written as (note that the moments and torque are unchanged "across" the cable attachment point)

$$\begin{bmatrix} V \\ M_v \\ \phi \\ v \\ W \\ M_w \\ \psi \\ w \\ F \\ u \\ T \\ \theta \end{bmatrix}_L = \begin{bmatrix} 0 & 0 & 0 & a_{1,4} & 0 & 0 & 0 & a_{1,8} & 0 & a_{1,10} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_{5,4} & 0 & 0 & 0 & a_{5,8} & 0 & a_{5,10} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_{9,4} & 0 & 0 & 0 & a_{9,8} & 0 & a_{9,10} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} V \\ M_v \\ \phi \\ v \\ W \\ M_w \\ \psi \\ w \\ F \\ u \\ T \\ \theta \end{bmatrix}_R$$

where

$$a_{1,4} = -[\sin^2\beta\cos^2\delta(k_c - P_c/L_c) + P_c/L_c]$$

$$a_{1,8} = \sin\beta\cos\beta\cos\delta(k_c - P_c/L_c)$$

$$a_{1,10} = \sin\delta\cos\delta\sin^2\beta(k_c - P_c/L_c)$$

$$a_{5,4} = a_{1,8}$$

$$a_{5,8} = -[\cos^2\beta(k_c - P_c/L_c) + P_c/L_c]$$

$$a_{5,10} = -\sin\beta\cos\beta\sin\delta(k_c - P_c/L_c)$$

$$a_{9,4} = \sin\delta\cos\delta\sin^2\beta(k_c - P_c/L_c)$$

$$a_{9,8} = a_{5,10}$$

$$a_{9,10} = -[\sin^2\beta\sin^2\delta(k_c - P_c/L_c) + P_c/L_c]$$

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